Available online at www.sciencedirect.com
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ELSEVIER
J. Math. Anal. Appl. 337 (2008) 1053–1063
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On a class of forced vibration problems with obstacles ☆

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Received 1 February 2007
Available online 27 April 2007
Submitted by Goong Chen

Abstract

The main purpose of this paper is to study the following damped vibration problems

$$-\ddot{x} = q(x)|\dot{x}|^2 + g(t)\dot{x} + f(t) \tag{1.1}$$

with

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = 0, \quad x(t) \geqslant 0,\ \forall t \in R \tag{1.2}$$

and

$$\dot{x}(t_0^-) = -\dot{x}(t_0^+), \quad \text{if } x(t_0) = 0, \tag{1.3}$$

where

$$\dot{x}(t_0^-) = \lim_{t\to t_0-0} \dot{x}(t), \qquad \dot{x}(t_0^+) = \lim_{t\to t_0+0} \dot{x}(t).$$

The variational sets are given and some multiplicity results of periodic solutions satisfying (1.1)–(1.3) are obtained.

Keywords: Second order Hamiltonian system; Periodic solution; Critical point

1. Introduction and preliminaries

Throughout the paper, we shall consider, as usual, $u = v$ means $u(t) = v(t)$ for a.e. $t \in R$. Let $f : R \to R$ and $q : [0, +\infty) \to [0, +\infty)$ be two continuous functions with

$$\lim_{x\to+\infty} q(x) = +\infty,$$

and $g : R \to R$ is continuous, $G(t) = \int_0^t g(s)\,ds$, $G(2\pi) = 0$. We look for the solutions of

$$-\ddot{x} = q(x)|\dot{x}|^2 + g(t)\dot{x} + f(t) \tag{1.1}$$

☆ The work is partially supported by the National Natural Science Foundation of China (10561011) and the Natural Science Foundation of Yunnan Province, China.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.04.036

satisfying the following conditions:

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = 0, \quad x(t) \geqslant 0, \ \forall t \in R, \tag{1.2}$$

$$\dot{x}(t_0^-) = -\dot{x}(t_0^+), \quad \text{if } x(t_0) = 0, \tag{1.3}$$

where

$$\dot{x}(t_0^-) = \lim_{t \to t_0 - 0} \dot{x}(t), \qquad \dot{x}(t_0^+) = \lim_{t \to t_0 + 0} \dot{x}(t).$$

Such a solution is called bouncing periodic solution of (1.1). Physically, it means that the particle bounces in a perfectly elastic way when it hits the obstacle $x = 0$.

As $q \equiv 0$ and $g \equiv 0$, the existence of the bouncing periodic solutions and quasiperiodic solutions of (1.1) has been considered by several authors in the last decade (see [1–3,7,8,10,13]). But, their methods are not variational. In 2005, M.Y. Jiang [5] take the lead in using the variational methods to study the existence of a sequence of periodic bouncing solutions for

$$-\ddot{x} = f(t, x).$$

Recently, Wu [12] studied the existence of a sequence of periodic bouncing solutions for

$$-\ddot{x} = \alpha(t)\dot{x} + f(t, x)$$

by the variational methods.

In the present paper, our purpose is to study the existence of a sequence of periodic bouncing solutions for

$$-\ddot{x} = q(x)|\dot{x}|^2 + g(t)\dot{x} + f(t)$$

by the variational methods. Both the problems and the results what we researching are new. It should be pointed that the problem (1.1) with (1.2) and (1.3) has no direct variational structure, generally.

We need the following preliminaries.

Let X be a real Banach space and X^* the dual space of X. A functional $J : X \to R$ is called locally Lipschitz if for each $u \in X$ there exist a neighborhood U of u and a constant $L \geqslant 0$ such that

$$\left|J(v) - J(w)\right| \leqslant L\|v - w\|, \quad \forall v, w \in U.$$

For any $u, v \in X$, we define the generalized directional derivative $J^0(u; v)$ of J at point u along the direction v as

$$J^0(u; v) = \overline{\lim_{h \to 0, \lambda \downarrow 0}} \frac{1}{\lambda}\left[J(u + h + \lambda v) - J(u + h)\right].$$

The generalized gradient of the function J at u, denoted by $\partial J(u)$, is the set

$$\partial J(u) = \left\{w \in X^*: \ \langle w, v\rangle \leqslant J^0(u; v), \ \forall v \in X\right\}.$$

Set

$$\lambda(u) = \min_{w \in \partial J(u)} \|w\|.$$

A point $u \in X$ is said to be a critical point of J if $\lambda(u) = 0$. Let X be a normed linear space and $f : X \to R$ a locally Lipschitz function. We say that f satisfies the (C) (or $(\mathrm{C})_c$) condition, if any sequence $\{x_n\} \subset X$ along which $f(x_n)$ is bounded (or $f(x_n) \to c$) and $(1 + \|x_n\|)\lambda(x_n) \to 0$ possesses a convergent subsequence. We say that f satisfies the $P.S$ condition, if any sequence $\{x_n\} \subset X$ along which $f(x_n)$ is bounded and $\lambda(x_n) \to 0$ possesses a convergent subsequenc.

In [6], the authors gave the following deformation theorem.

Theorem 1.1. *Let X be a reflexive Banach space and $f : X \to R$ a locally Lipschitz function with the condition* (C) *in $f^{-1}((a, b))$. Then for any $c \in (a, b)$, any $\varepsilon_0 > 0$ and any neighborhood N of K_c, there exist $\varepsilon \in (0, \varepsilon_0)$ and a continuous mapping $\eta : [0.1] \times X \to X$ such that for all $(t, x) \in [0.1] \times X$ we have*

(a) $\|\eta(t, x) - x\| \leqslant e(1 + \|x\|)t$*, where e is a constant*;

(b) $|f(x)-c| \geqslant \varepsilon_0 \Rightarrow \eta(t,x)=x$;
(c) $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon} \cup N$;
(d) *$f(\eta(t,x))$ is nonincreasing in t*;
(e) *$\eta(t,\cdot): X \to X$ is a homeomorphism*;
(f) $\eta(t,x) \neq x \Rightarrow f(\eta(t,x)) < f(x)$.

Using the deformation theorem, as Theorem 9.12 in [11] we can prove the following nonsmooth version of symmetric mountain pass theorem.

Theorem 1.2. *Let E be a reflexive infinite dimensional Banach space and let $I: E \to R$ be an even locally Lipschitz function with the condition* (C) *and $I(0)=0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies*

(i) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geqslant \alpha$; and*
(ii) *for each finite dimensional subspace $\tilde{E} \subset E$, there is an $r = r(\tilde{E})$ such that $I \leqslant 0$ on $\tilde{E} \backslash B_r$,*

then I possesses an unbounded sequence of critical values, where $B_r = \{x \in E: \|x\| < r\}$.

2. The variational setting

We set

$$Q(s) = \int_0^s q(r)\,dr, \qquad y = \Psi(x) := \int_0^x e^{Q(s)}\,ds.$$

Then the problem (1.1) with (1.2) and (1.3) is equivalent to

$$-\ddot{y} = g(t)\dot{y} + f(t)\big[1 + h(y)\big] \tag{2.1}$$

satisfying the following conditions

$$y(0) - y(2\pi) = \dot{y}(0) - \dot{y}(2\pi) = 0, \quad y(t) \geqslant 0,\ \forall t \in R, \tag{2.2}$$

$$\dot{y}\big(t_0^-\big) = -\dot{y}\big(t_0^+\big), \quad \text{if } y(t_0) = 0, \tag{2.3}$$

where

$$h(y) = e^{Q(\Psi^{-1}(y))} - 1 = \int_0^y q\big(\Psi^{-1}(s)\big)\,ds$$

with the following properties:

(1) $h(0)=0$, $h: [0,+\infty) \to [0,+\infty)$ is differentiable and increasing;
(2) $\lim_{r\to 0} \frac{h(r)}{r} = h'(0) = q(0)$;
(3) $\lim_{r\to +\infty} \frac{h(r)}{r} = +\infty$;
(4) $\int_0^{+\infty} \frac{1}{1+h(r)}\,dr = +\infty$.

Obviously, we have the following proposition.

Proposition 2.1. *If $\tilde{y}$ is a solution of*

$$-\ddot{y} = g(t)\dot{y} + \big[f(t)\big(1 + h\big(|y|\big)\big)\big]\operatorname{sgn}(y) \tag{2.4}$$

satisfying the periodic boundary condition

$$\tilde{y}(0) - \tilde{y}(2\pi) = \dot{\tilde{y}}(0) - \dot{\tilde{y}}(2\pi) = 0 \tag{2.5}$$

or the anti-periodic boundary condition

$$\tilde{y}(0) = -\tilde{y}(2\pi), \qquad \dot{\tilde{y}}(0) = -\dot{\tilde{y}}(2\pi), \tag{2.6}$$

then $y = |\tilde{y}|$ *is a bouncing* 2π*-periodic solution of* (2.1).

We will state the variational principle for periodic solutions of (2.4) (see [12]).

Let $E = H^1_{2\pi} = \{x : [0, 2\pi] \to R\colon\ x \text{ is absolutely continuous},\ x(0) = x(2\pi),\ \dot{x} \in L^2(0, 2\pi)\}$. Then E is a Hilbert space with the norm

$$\|x\| = \left[\int_0^{2\pi} \left|x(t)\right|^2 dt + \int_0^{2\pi} \left|\dot{x}(t)\right|^2 dt\right]^{\frac{1}{2}}.$$

Define the functional I on E, given by

$$I(y) = \frac{1}{2}\int_0^{2\pi} e^{G(t)} \dot{y}^2 \, dt - \int_0^{2\pi} e^{G(t)} f(t)\bigl(|y| + H\bigl(|y|\bigr)\bigr) \, dt$$

where $H(|y|) = \int_0^{|y|} h(s)\, ds$. The functional I is locally Lipschitz on E. But it may be not continuously differentiable on E.

Theorem 2.2. *Let u be a critical point of the functional I on E.*

(i) *If all zero points of u are isolated, then u is a 2π-periodic solutions of* (2.4) *with* (2.5).

(ii) *If there exists a $t_0 \in [0, 2\pi]$ such that $f(t_0) > 0$ and $u(t_0) = \dot{u}(t_0) = 0$, then $u = 0$ on σ_0, where σ_0 is the connect component of the set $\{t \in [0, 2\pi] : f(t) > 0\}$ containing t_0. Particularly, if $u \neq 0$ on σ_0, then the zeros of u in σ_0 are isolated.*

(iii) *If the following conditions hold*:
 (a) *$f(t) = 0$, for a.e. $t \in [0, 2\pi]$,*
 (b) *there exists $t_0 \in [0, 2\pi]$ such that $u(t_0) = \dot{u}(t_0) = 0$,*
 then $u = 0$ on $[0, 2\pi]$. Particularly, if $u \neq 0$, then the zeros of u in $[0, 2\pi]$ are isolated.

(iv) *If the following conditions hold*:
 (a) *$f(t) > 0$, for a.e. $t \in [0, 2\pi]$,*
 (b) *there exists $t_0 \in [0, 2\pi]$ such that $u(t_0) = \dot{u}(t_0) = 0$,*
 then $u = 0$ on $[0, 2\pi]$. Particularly, if $u \neq 0$, then the zeros of u in $[0, 2\pi]$ are isolated.

Proof. Since $u \in H^1_{2\pi}$ is a critical point of I, $0 \in \partial I(u)$. Set $J(u) = \int_0^{2\pi} e^{G(t)} f(t)(|u(t)| + H(|u(t)|))\, dt$. Then

$$\partial J(u) \subset e^{G(t)}\bigl[\underline{f}\bigl(t, \bigl|u(t)\bigr|\bigr), \overline{f}\bigl(t, \bigl|u(t)\bigr|\bigr)\bigr], \quad \text{a.e. } t \in [0, 2\pi],$$

where

$$\underline{f}(t, s) = \min\Bigl\{\lim_{\tau \to s-0} f(t)\bigl(1 + h\bigl(|s|\bigr)\bigr)\operatorname{sgn}(\tau),\ \lim_{\tau \to s+0} f(t)\bigl(1 + h\bigl(|s|\bigr)\bigr)\operatorname{sgn}(\tau)\Bigr\}$$

and

$$\overline{f}(t, s) = \max\Bigl\{\lim_{\tau \to s-0} f(t)\bigl(1 + h\bigl(|s|\bigr)\bigr)\operatorname{sgn}(\tau),\ \lim_{\tau \to s+0} f(t)\bigl(1 + h\bigl(|s|\bigr)\bigr)\operatorname{sgn}(\tau)\Bigr\}.$$

Hence there exists a function $\xi(t) \in [\underline{f}(t, |u(t)|), \overline{f}(t, |u(t)|)]$ such that

$$\int_0^{2\pi} e^{G(t)}\bigl[\dot{u}(t)\dot{v}(t) - \xi(t)v(t)\bigr]\, dt = 0$$

for every $v \in H^1_{2\pi}$. By Fundamental Lemma and remarks in [9, pp. 6–7] we know that $e^{G(t)}\dot{u}(t)$ has a weak derivative, and

$$\big[e^{G(t)}\dot{u}(t)\big]' = -e^{G(t)}\xi(t), \quad \text{a.e. on } \in [0, 2\pi], \tag{2.7}$$

$$e^{G(t)}\dot{u}(t) = -\int_0^t e^{G(s)}\xi(s)\,ds + c, \quad \text{a.e. on } \in [0, 2\pi], \tag{2.8}$$

$$-\int_0^{2\pi} e^{G(t)}\xi(t)\,dt = 0, \tag{2.9}$$

where c is a constant. We identify the equivalence class $e^{G(t)}\dot{u}(t)$ and its continuous representant $\int_0^t -e^{G(s)}\xi(s)\,ds + c$. Then $\dot{u}$ is absolutely continuous, and by (2.8), (2.9), one has

$$\dot{u}(0) - \dot{u}(2\pi) = u(0) - u(2\pi) = 0.$$

(i) Notice that all zero points of u are isolated. By (2.7) we know

$$-\ddot{u}(t) = g(t)\dot{u}(t) + f(t)\big(1 + h\big(|u(t)|\big)\big)\operatorname{sgn}\big(u(t)\big), \quad \text{a.e. } t \in [0, 2\pi].$$

Hence u is a 2π-periodic solution of (2.4) with (2.5).

(ii) Let $\sigma_1 = \{t \in \sigma_0\colon\ u(t) = \dot{u}(t) = 0\}$. Then $\sigma_1 \neq \emptyset$. For each $t^* \in \sigma_1$ and for each t near t^* we can assume $t > t^*$. Set

$$K(t) = \frac{1}{2}e^{G(t)}\big|\dot{u}(t)\big|^2 + e^{G(t)}f(t^*)\big|u(t)\big|.$$

Then $K(t) \geqslant 0$.

On the other hand, by (2.7) one has

$$\begin{aligned}
K(t) - e^{G(t)}f(t^*)\big|u(t)\big| &= \frac{1}{2}e^{G(t)}\big|\dot{u}(t)\big|^2 \\
&= \int_{t^*}^t \left[\frac{1}{2}e^{G(s)}\big|\dot{u}(s)\big|^2\right]' ds \\
&= \int_{t^*}^t \left[\frac{1}{2}g(s)e^{G(s)}\big|\dot{u}(s)\big|^2 + e^{G(s)}\ddot{u}(s)\dot{u}(s)\right] ds \\
&= -\frac{1}{2}\int_{t^*}^t g(s)e^{G(s)}\big|\dot{u}(s)\big|^2\,ds - \int_{t^*}^t e^{G(s)}\dot{u}(s)\xi(s)\,ds \\
&\leqslant C_1\int_{t^*}^t \frac{1}{2}e^{G(s)}\big|\dot{u}(s)\big|^2\,ds - \int_{t^*}^t e^{G(s)}\dot{u}(s)\xi(s)\,ds.
\end{aligned}$$

By the continuity of f and g, there is constant $C_2 > 0$ such that

$$K(t) \leqslant C_1\int_{t^*}^t K(s)\,ds + C_2.$$

Whenever $\int_{t^*}^t e^{G(s)}|\dot{u}(s)|^2\,ds > 0$, there is constant $C > 0$ such that

$$K(t) \leqslant C\int_{t^*}^t K(s)\,ds.$$

By Gronwall inequality, one has $K(t) \leqslant 0$ for t near t^*. Hence $K(t) = 0$ for t near t^*, so that $u(t) = u(t^*) = 0$ for t near t^*, a contradiction. Hence $\int_{t^*}^{t} e^{G(s)}|\dot{u}(s)|^2\,ds = 0$. This implies $\dot{u}(t) = 0$ and so $u(t) = u(t^*) = 0$ for t near t^*. This shows σ is a nonempty open set of $[0, 2\pi]$. Moreover, obviously, σ is closed set of $[0, 2\pi]$. Hence $\sigma = [0, 2\pi]$, and hence $u = 0$.

If $u \neq 0$ and the zeros of u in $[0, 2\pi]$ are not isolated, we can assume that t_0 is not a isolated zero, then there exists $\{t_n\}$ with $u(t_n) = 0$ and $t_n \to t_0$. By the definition of derivative we have $\dot{u}(t_0) = 0$. So $u = 0$, a contradiction. Hence the zeros of u in $[0, 2\pi]$ are isolated.

(iii) If

$$f(t) = 0 \quad \text{for a.e. } t \in [0, 2\pi],$$

then $\int_0^{2\pi} |\dot{u}(t)|^2\,dt = 0$, and hence

$$\dot{u}(t) = 0 \quad \text{for a.e. } t \in [0, 2\pi].$$

Hence, by the absolute continuity of u and $u(t_0) = 0$, $u(t) \equiv 0$ on $[0, 2\pi]$.

(iv) From the proofs of (ii) we know that

$$\dot{u}(t) = 0 \quad \text{for a.e. } t \in [0, 2\pi].$$

Hence, by the absolute continuity of u and $u(t_0) = 0$, $u(t) \equiv 0$ on $[0, 2\pi]$. □

3. Existence of a sequence of periodic bouncing solutions

Theorem 3.1. *If $f(t) > 0$ for all $t \in [0, 2\pi]$ and there are constants $\theta > 2$, $r > 0$ such that*

$$xh(x) - \theta H(x) \geqslant 0, \quad \forall x \geqslant r, \tag{h_1}$$

then (1.1) *has an unbounded sequence of 2π-periodic solutions $\{u_j\}$ satisfying* (1.2) *and* (1.3).

Proof. Let X_2 be a finite dimensional subspace of E given by

$$X_2 = \left\{ \sum_{j=0}^{k_0} (a_j \cos jt + b_j \sin jt) \,\middle|\, a_j, b_j \in R,\ j = 0, \ldots, k_0 \right\}$$

and let $X_1 = X_2^{\perp}$. Then $E = X_1 \oplus X_2$. It is obvious that we have

$$\|\dot{x}\|_2^2 \leqslant k_0^2 \|x\|_2^2 \quad \forall x \in X_2.$$
$$\|\dot{x}\|_2^2 \geqslant (k_0+1)^2 \|x\|_2^2 \quad \forall x \in X_1.$$

Set $d_1 = \min_{t\in[0,2\pi]} e^{G(t)}$, $d_2 = \max_{t\in[0,2\pi]} e^{G(t)}$. Then $0 < d_1 \leqslant d_2 < +\infty$. By the continuity of f, g and h we know that for $|x| \leqslant 1$, there is a constant $M_1 > 0$ such that $|f(t)e^{G(t)}(|x| + H(|x|))| \leqslant M_1|x|$ for all $t \in [0, 2\pi]$. Consequently, for $u \in X_1$ with small $\|u\|$, one has

$$I(u) \geqslant \frac{d_1}{2} \int_0^{2\pi} \left|\dot{u}(t)\right|^2 dt - M_1 \int_0^{2\pi} \left|u(t)\right| dt \geqslant \frac{d_1}{4}\|u\|^2 - \frac{\sqrt{2\pi}}{k+1} M_1 \|u\|.$$

Take a small $\rho > 0$ and an integer $k_0 > \frac{4\sqrt{2\pi}M_1}{d_1\rho} + 1$. Then for $u \in X_1$ with $\|u\| = \rho$, one has $I(u) \geqslant \frac{d_1}{4}\rho^2 - \frac{\sqrt{2\pi}}{k_0+1}M\rho = \alpha_{k_0} > 0$.

For any finite dimensional subspace $X \subset E$, there is a large integer k such that $X \subset H_k := \{\sum_{j=0}^{k}(a_j \cos jt + b_j \sin jt) \mid a_j, b_j \in R,\ j = 0, \ldots, k\}$. Take large $M > 0$ be such that $Md_1 > d_2k^2$. By the property (3) of (h) and the continuity of f, there is an $r_1 > 0$ such that $f(t)(1 + h(\xi)) \geqslant M\xi$ for all $\xi \geqslant r_1$ and a.e. $t \in [0, 2\pi]$. Hence,

$$f(t)\bigl(1 + h(\xi)\bigr) \geqslant M\xi - Mr_1, \quad \forall \xi \in [0, +\infty) \text{ and a.e. } t \in [0, 2\pi],$$

and hence

$$f(t)\bigl(|x| + H\bigl(|x|\bigr)\bigr) \geqslant \frac{1}{2}M|x|^2 - Mr_1|x|, \quad \forall x \in R \text{ and a.e. } t \in [0, 2\pi].$$

Consequently, for any $u \in X$, one has

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_0^{2\pi} e^{G(t)}\big|\dot{u}(t)\big|^2\,dt - \int_0^{2\pi} e^{G(t)} f(t)\big(\big|u(t)\big| + H\big(\big|u(t)\big|\big)\big) \\
&\leqslant \frac{d_2}{2}\int_0^{2\pi}\big|\dot{u}(t)\big|^2\,dt - d_1\int_0^{2\pi}\bigg[\frac{1}{2}M\big|u(t)\big|^2 - Mr_1\big|u(t)\big|\bigg]\,dt \\
&\leqslant \frac{d_2}{2}k^2\|u\|_2^2 - \frac{d_1}{2}M\|u\|_2^2 + Md_1r_1\|u\|_1 \\
&= \frac{1}{2}\big(d_2k^2 - d_1M\big)\|u\|_2^2 + Md_1r_1\|u\|_1.
\end{aligned}$$

Since all norms are equivalent in a finite dimensional space, there is an $L = L(X) > 0$ such that $I(u) < 0$ for all $u \in X$ with $\|u\| \geqslant L$.

Now, we prove I satisfies the (C) condition.

Assume $\{x_n\} \subset E$ with $\{I(x_n)\}$ is bounded and $(1 + \|x_n\|)\lambda(x_n) \to 0$ as $n \to +\infty$, where $\lambda(x_n) = \min_{\eta\in\partial I(x_n)}\|\eta\|$. Choose $\eta_n \in \partial I(x_n)$ be such that $\|\eta_n\| = \lambda(x_n)$. Then there exists $y_n \in [\underline{f}(t, |x_n(t)|), \overline{f}(t, |x_n(t)|)]$ such that

$$\langle \eta_n, y\rangle = \int_0^{2\pi} e^{G(t)}\dot{x}_n(t)\dot{y}(t)\,dt - \int_0^{2\pi} e^{G(t)} y_n(t) y(t)\,dt, \quad \forall y \in E.$$

We claim that the sequence $\{x_n\}$ is bounded. Otherwise, we can assume $\|x_n\| \to \infty$. Set $\omega_n = \frac{x_n}{\|x_n\|}$. Then $\{\omega_n\}$ is bounded in E. Since $H^1_{2\pi}$ is a Hilbert space, we can assume that there exists $\omega \in H^1_{2\pi}$ such that

$$\omega_n \rightharpoonup \omega \quad \text{in } H^1_{2\pi},$$

and hence $\{\omega_n\}$ converges uniformly to ω on $[0, 2\pi]$ by Proposition 1.2 in [11]. Set $\Omega = \{t \in [0, 2\pi] : \omega(t) \neq 0\}$. If the measure $|\Omega| \neq 0$, then $|x_n(t)| \to \infty$ for a.e. $t \in \Omega$, and hence

$$\begin{aligned}
d_2 &\geqslant \lim_{n\to\infty}\frac{1}{\|x_n\|^2}\int_0^{2\pi}\big[e^{G(t)}y_n(t)x_n(t) + d_2\big|x_n(t)\big|^2\big]\,dt \\
&\geqslant \lim_{n\to\infty}\int_\Omega d_1\bigg[\frac{f(t)h(|x_n|)}{|x_n|} + 1\bigg]\omega_n^2\,dt \\
&= +\infty.
\end{aligned}$$

This is a contradiction. Hence $|\Omega| = 0$, namely $\omega(t) = 0$, a.e. $t \in [0, 2\pi]$.

Moreover, by (h_1), one has

$$\begin{aligned}
\theta I(x_n) - \langle \omega_n, x_n\rangle &= \bigg(\frac{\theta}{2} - 1\bigg)\int_0^{2\pi} e^{G(t)}\big|\dot{x}_n(t)\big|^2\,dt + \int_0^{2\pi} e^{G(t)}\big[y_n(t)x_n(t) - \theta f(t)\big(\big|x_n(t)\big| + H\big(\big|x_n(t)\big|\big)\big)\big]\,dt \\
&= \bigg(\frac{\theta}{2} - 1\bigg)\int_0^{2\pi} e^{G(t)}\big|\dot{x}_n(t)\big|^2\,dt + \int_0^{2\pi} e^{G(t)} f(t)(1-\theta)\big|x_n(t)\big|\,dt \\
&\quad + \int_0^{2\pi} e^{G(t)} f(t)\big[\big|x_n(t)\big|h\big(\big|x_n(t)\big|\big) - \theta H\big(\big|x_n(t)\big|\big)\big]\,dt
\end{aligned}$$

$$\geqslant \left(\frac{\theta}{2}-1\right)d_1\int_0^{2\pi}\left|\dot{x}_n(t)\right|^2dt-\int_0^{2\pi}e^{G(t)}f(t)\max_{|u|\leqslant r}\left[h\left(|u|\right)|u|-\theta H\left(|u|\right)\right]dt$$

$$+\int_0^{2\pi}e^{G(t)}f(t)(1-\theta)\left|x_n(t)\right|dt$$

$$=\left(\frac{\theta}{2}-1\right)d_1\int_0^{2\pi}\left|\dot{x}_n(t)\right|^2dt+\int_0^{2\pi}e^{G(t)}f(t)(1-\theta)\left|x_n(t)\right|dt-c_6.$$

This implies that $\int_0^{2\pi}|\dot{\omega}_n(t)|^2\,dt\to 0$ as $n\to\infty$. Consequently,

$$1=\int_0^{2\pi}\left|\dot{\omega}_n(t)\right|^2dt+\int_0^{2\pi}\left|\omega_n(t)\right|^2dt\to 0,$$

a contradiction. Hence $\{x_n\}$ is bounded in E. As the proof 2° of Theorem 4.3 in [4] we may prove that $\{x_n\}$ possess a convergent subsequence. So I satisfies condition (C).

Obviously, I is even and $I(0)=0$. By virtue of Theorem 1.2, I has a sequence of critical points $\{y_j\}$ on E such that $|I(y_j)|\to+\infty$. If $\{y_j\}$ is bounded in E, then by the definition of I, one know that $\{I(y_j)\}$ is also bounded, a contradiction. Hence $\{y_j\}$ is unbounded in E. By Theorem 2.2 and Proposition 2.1, $\{|y_j|\}$ is an unbounded sequence of bouncing 2π-periodic solution of (2.1). Set $x_j=\Psi^{-1}(|y_j|)$. Then $\{x_j\}$ is an unbounded sequence of bouncing 2π-periodic solution of (1.1). □

Theorem 3.2. *If $f(t)>0$ for all $t\in[0,2\pi]$*

$$\lim_{x\to+\infty}\frac{q(x)\int_0^x e^{Q(r)}\,dr}{e^{Q(x)}-1}\geqslant\alpha>1, \qquad (q_1)$$

then (1.1) *has an unbounded sequence of 2π-periodic solutions $\{u_j\}$ satisfying* (1.2) *and* (1.3).

Proof. Since

$$\begin{aligned}\lim_{x\to+\infty}\frac{xh(x)}{H(x)}&=\lim_{x\to+\infty}\frac{h(x)+xq(\Psi^{-1}(x))}{h(x)}\\&=1+\lim_{x\to+\infty}\frac{xq(\Psi^{-1}(x))}{\int_0^x q(\Psi^{-1}(s))\,ds}\\&=1+\lim_{x\to+\infty}\frac{xq(\Psi^{-1}(x))}{\int_0^{\Psi^{-1}(x)}q(r)e^{Q(r)}\,dr}\\&=1+\lim_{x\to+\infty}\frac{q(x)\int_0^x e^{Q(r)}\,dr}{\int_0^x q(r)e^{Q(r)}\,dr}\\&=1+\lim_{x\to+\infty}\frac{q(x)\int_0^x e^{Q(r)}\,dr}{e^{Q(x)}-1}\\&\geqslant 1+\alpha.\end{aligned}$$

Consequently, the condition (h_1) of Theorem 3.1 holds. Hence the conclusion follows from Theorem 3.1. □

Remark 3.1. Since the condition (q_1) is, indeed, a variant of Ambrosetti–Rabinowitz-type condition (see (p_4) in [11, p. 9]), there are functions satisfying all conditions of Theorem 3.2.

Theorem 3.3. *If $f(t) > 0$ for all $t \in [0, 2\pi]$ and there exist constants $r \in (2, +\infty)$ and $\mu \in [1, +\infty)$ with $\mu > r - 2$ such that*

$$\limsup_{\xi \to +\infty} \frac{h(\xi)}{\xi^{r-1}} < +\infty \tag{h_2}$$

and

$$\liminf_{\xi \to +\infty} \frac{\xi h(\xi) - 2H(\xi)}{\xi^{\mu}} > 0 \tag{h_3}$$

hold, then (1.1) *has an unbounded sequence of* 2π*-periodic solutions* $\{u_j\}$ *satisfying* (1.2) *and* (1.3).

Proof. It is sufficient to prove I satisfies the (C) condition from the proof of Theorem 3.1. Assume $\{x_n\} \subset E$ with $\{I(x_n)\}$ is bounded and $(1 + \|x_n\|)\lambda(x_n) \to 0$ as $n \to +\infty$, where $\lambda(x_n) = \min_{\omega \in \partial I(x_n)} \|\omega\|$. Choose $x_n^* \in \partial I(x_n)$ be such that $\|x_n^*\| = \lambda(x_n)$. Then there exists $z_n \in [\underline{f}(t, |x_n(t)|), \overline{f}(t, |x_n(t)|)]$ such that

$$\langle x_n^*, y \rangle = \int_0^{2\pi} e^{G(t)} \dot{x}_n(t) \dot{y}(t)\, dt - \int_0^{2\pi} e^{G(t)} z_n(t) y(t)\, dt, \quad \forall y \in E.$$

Hence

$$\langle x_n^*, x_n \rangle = \int_0^{2\pi} e^{G(t)} \big|\dot{x}_n(t)\big|^2\, dt - \int_0^{2\pi} e^{G(t)} z_n(t) x_n(t)\, dt.$$

We need only to prove that the sequence $\{x_n\}$ is bounded. Notice that

$$\begin{aligned} \frac{1}{2}\int_0^{2\pi} e^{G(t)} \big|\dot{x}_n(t)\big|^2\, dt &= \langle x_n^*, x_n \rangle - I(x_n) + \int_0^{2\pi} e^{G(t)} \big[z_n(t)x_n(t) - f(t)\big(\big|x_n(t)\big| + H\big(\big|x_n(t)\big|\big)\big)\big]\, dt \\ &\leqslant C_1 + \int_0^{2\pi} e^{G(t)} f(t) \big[\big|x_n(t)\big| h\big(\big|x_n(t)\big|\big) - H\big(\big|x_n(t)\big|\big)\big]\, dt, \end{aligned}$$

where $C_1 > 0$. By (h_2), for any fixed $\varepsilon > 0$, there exists $s_0 > 0$ such that

$$f(t)h(\xi) \leqslant \varepsilon \xi^{r-1}, \quad \forall \xi > s_0.$$

Since f and h are continuous, $M = \sup_{(t,\xi) \in [0,2\pi] \times [0,s_0]} |f(t)h(\xi)|$ is finite, and hence

$$f(t)h(\xi) \leqslant \varepsilon \xi^{r-1} + M$$

for all $(t, \xi) \in [0, 2\pi] \times [0, +\infty)$. Therefore,

$$f(t)H\big(|x|\big) = \int_0^{|x|} f(t)h(\xi)\, d\xi \leqslant \int_0^{|x|} \big(\varepsilon \xi^{r-1} + M\big)\, d\xi = \frac{\varepsilon}{r}|x|^r + M|x|.$$

Consequently,

$$\frac{1}{2}\|x_n\|^2 = \frac{1}{2}\int_0^{2\pi} \big[\big|\dot{x}_n(t)\big|^2 + \big|x_n(t)\big|^2\big]\, dt \leqslant C_2 + C_3\|x_n\|_{L^r}^r + C_4\|x_n\|_{L^r}^2, \tag{3.1}$$

where $C_2, C_3, C_4 > 0$. Moreover, by (h_3) and the Hölder inequality, one has

$$\begin{aligned}
2I(x_n)-\langle x_n^*, x_n\rangle &= \int_0^{2\pi} e^{G(t)} f(t)\big[\big(h(|x_n|)-1\big)|x_n|-2H(|x_n|)\big]\,dt \\
&= \int_0^{2\pi} e^{G(t)} f(t)\big[h(|x_n|)|x_n|-2H(|x_n|)\big]\,dt-\int_0^{2\pi} e^{G(t)} f(t)\big|x_n(t)\big|\,dt \\
&\geqslant C_5\|x_n\|_{L^\mu}^{\mu}-C_6\|x_n\|_{L^\mu}-C_7,
\end{aligned} \tag{3.2}$$

where $C_5, C_6, C_7>0$ are constants, and hence $\int_0^{2\pi}|x_n(t)|^\mu\,dt$ is bounded. If $\mu\geqslant r$, then by (3.1) we know that $\|x_n\|$ is bounded. If $\mu<r$, then by (3.1) and Proposition 1.1 in [9] and notice that $\mu>r-2$, we know that $\|x_n\|$ is bounded, too. □

Theorem 3.4. *Let $f(t)>0$ for all $t\in[0,2\pi]$. If there exist constants $\alpha\in(0,1)$ such that*

$$\lim_{\xi\to+\infty}\frac{q(\xi)}{e^{\alpha Q(\frac{\xi}{2})}}<+\infty \tag{q_2}$$

and

$$\lim_{\xi\to+\infty}\left[q(\xi)\int_0^{\xi} e^{Q(t)}\,dt-e^{Q(\xi)}\right]>0 \tag{q_3}$$

holds, then (1.1) *has an unbounded sequence of 2π-periodic solutions $\{u_j\}$ satisfying* (1.2) *and* (1.3).

Proof. Take $r=2+\alpha$, $\mu=1$. Then $r>2$ and $\mu>\alpha=r-2$. Then, by (q_2) and (q_3), one has

$$\begin{aligned}
\lim_{\xi\to+\infty}\frac{h(\xi)}{\xi^{r-1}} &= \lim_{\xi\to+\infty}\frac{h(\xi)}{\xi^{1+\alpha}} \\
&= \lim_{\xi\to+\infty}\frac{q(\Psi^{-1}(\xi))}{(1+\alpha)\xi^\alpha} \\
&= \lim_{s\to+\infty}\frac{q(s)}{(1+\alpha)\Psi^\alpha(s)} \\
&= \lim_{s\to+\infty}\frac{q(s)}{(1+\alpha)(\int_0^s e^{Q(t)}\,dt)^\alpha} \\
&\leqslant \lim_{s\to+\infty}\frac{q(s)}{(1+\alpha)(\int_{\frac{s}{2}}^s e^{Q(t)}\,dt)^\alpha} \\
&\leqslant \lim_{s\to+\infty}\frac{q(s)}{(1+\alpha)(\frac{s}{2})^\alpha e^{\alpha Q(\frac{s}{2})}} \\
&\leqslant \lim_{s\to+\infty}\frac{q(s)}{e^{\alpha Q(\frac{s}{2})}} \\
&< +\infty
\end{aligned}$$

and

$$\lim_{\xi\to+\infty}\frac{\xi h(\xi)-2H(\xi)}{\xi^\mu}=\lim_{\xi\to+\infty}\frac{\xi q(\Psi^{-1}(\xi))-h(\xi)}{\mu\xi^{\mu-1}}=\lim_{\xi\to+\infty}\left[q(\xi)\int_0^{\xi} e^{Q(t)}\,dt-e^{Q(\xi)}\right]>0.$$

Consequently, the conclusion follows from Theorem 3.3. □

Remark 3.2. Take $q(x) = \ln(1 + x)$. Then the conditions (q_2) and (q_3) are satisfied.

References

[1] D. Bonheure, C. Fabry, Periodic motions in impact oscillators with perfectly elastic bounces, Nonlinearity 15 (2002) 1281–1297.
[2] C.J. Budd, The global dynamics of impact oscillators, in: Real and Complex Dynamical Systems, NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci. 644 (1995) 27–46.
[3] C.J. Budd, F. Dux, Intermittency in impact oscillators close to resonance, Nonlinearity 7 (1994) 1191–1224.
[4] K.-C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1981) 102–129.
[5] M.Y. Jiang, Periodic solutions of second order Hamiltonian systems with an obstacle, a report of the 14th Nationwide Nonlinear Functional Analysis Conference in China.
[6] N.C. Kourogenis, N.S. Papageorgiou, Nonsmooth critical point theory and nonlinear elliptic equations at resonance, J. Aust. Math. Soc. 69 (2000) 245–271.
[7] H. Lamba, Chaotic regular and unbounded behavior in the elastic impact oscillator, Phys. D 82 (1995) 117–135.
[8] A.C. Lazer, P.J. Mckenna, Periodic bouncing for a forced linear spring with obstacle, Differential Integral Equations 5 (1992) 165–172.
[9] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York/Berlin, 1989.
[10] P. Prtega, Dynamics of a Forced Oscillator with Obstacle, Variational and Topological Methods in the Study of Nonlinear Phenomena, Birkhäuser, Boston–Basel–Berlin, 2002, pp. 75–87.
[11] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[12] X. Wu, On a class of damped vibration problems with obstacles, J. Differential Equations, in press.
[13] V. Zharnitsky, Invariant tori in Hamiltonian systems with impacts, Comm. Math. Phys. 211 (2000) 289–302.